Design of Adaptive Systems by Using Fractional Calculus System's Approach

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Abstract: The Model Reference Control System (MRACS) is the control system tuning its controller to match the controlled system to the reference model designed in the\system adaptively. Conventional MRACS assumed the order of integration or derivative as integer order. By introducing fractional order integration or derivative (Fractional Calculus,) MRACS can use the arbitrary order adaptive control law and control the complex system described by fractional calculus like visco-elastic body. In this paper, we improved the transient response of MRACS and modified MRACS to be able to control the plant even though the model matching condition was not satisfied by applying fractional calculus to adaptive transfer function of MRACS, and we constructed the MRACS for the fractional calculus system.

Key Words: Adaptive control, Fractional calculus, Fractional order control, Model reference control system

1 INTRODUCTION

Recently, many researchers study the control theory applying fractional calculus (fractional order derivative or integration) to the controller or the controlled object.[1],[2] Fractional calculus can describe many complicated systems which can not be described simply with conventional integer order derivative or integration.[3] In addition, fractional calculus can construct the fractional order integrator. So, by using fractional calculus system in controller, we can design the new control law containing arbitrary order integration.

Model Reference Control System (MRACS) is the control system tuning its controller to match the controlled system to the designed reference model if plant parameters are unknown, but conventional MRACS don't consider the system described by fractional order derivative. Inaddition, to design the MRACS, the assumption of the relative order of the controlled object is needed in order to stabilize the total adaptive control systems. If we design the MRACS without the assumption for the relative order of the controlled object, we must use augmented error method, high order estimator, Back stepping, or I&I.

In this paper, by introducing fractional calculus system to the adaptive control law, we can get the stable numerical results whose transient performance is good in the case of lack of the model matching condition and improve the transient response of controlled object. In addition, by considering the MRACS extended to fractional order, we designed MRACS for the fractional calculus system.

2 FRACTIONAL CALCULUS

2.1 Fractional order derivative

The fractional order Rieman-Liouville derivative Eq. (1) is given by

$$\frac{\mathrm{d}^q}{\mathrm{d}t^q} f(t) \equiv {}_aD_t^q[f(t)] = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t \frac{(t-\tau)^{n-q-1}}{\Gamma(n-q)} f(\tau)\mathrm{d}\tau, \qquad (1)$$

$$\Gamma(x) := \int_0^\infty \exp(-t)t^{x-1}\mathrm{d}t, \quad x(\in \mathrm{R}) > 0, \qquad (2)$$

where q is the order of the fractional derivative such that $n-1<q<n$, n is the integer, and $\Gamma(x)$ is the gamma function, which is the function expanding the factorial to arbitraly order.

Considering the following function to show the validness of the definition of Eq. (1) and Eq. (2),

$$f_1(t) = \sin t, \quad -\infty < t < \infty \qquad (3)$$

$$f_2(t) = \begin{cases} t^2 & 0 \le t \\ 0 & t < 0 \end{cases} \qquad (4)$$

${}_{-\infty}D_t^q[f_1(t)]$ and ${}_{-\infty}D_t^q[f_2(t)]$ were shown in Fig. 1 and Fig. 2 with changing the order of the fractional derivative q.

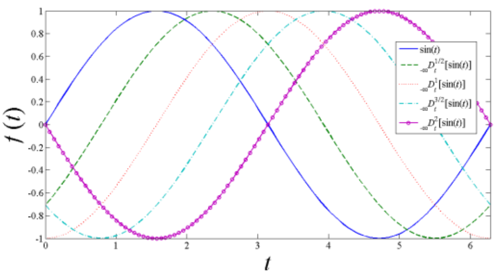

Fig. 1 The function of ${}_{-\infty}D_t^q[f_1(t)]$.

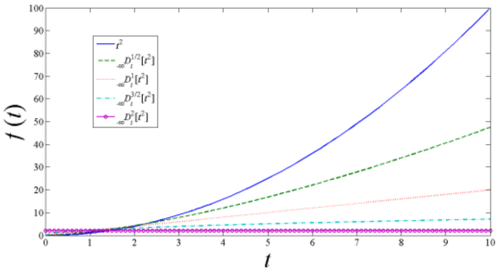

Fig. 2 The function of ${}_{-\infty}D_t^q[f_2(t)]$.

In Fig. 1, the fractional order Rieman-Liouville derivative expresses the phase shift derived from the derivation of the sinusoidal function analogically. This feature has the important meaning. By Fourier transform, any function can be converted to the frequency domain spectrum. Considering the derivative as the operation to shift the sinusoidal functions' phase with multiplying its angular frequency, it means that any function can be analogically differentiated by the fractional order Rieman-Liouville derivative. As shown in Fig. 2, the fractional order Rieman-Liouville derivative expresses analogical derivative of $f_2(t)$ though it is not sinusoidal function.

2.2 Approximation method of the fractional order transfer function

In the case of constructing the control system using the fractional order transfer function, it requires a lot of time to calculate the covolve from the initial time.

For reducing the time on calculation, Manabe proposed the approach to approximate the fractional order transfer function to the superposition of the integer order transfer functions on the bode diagram around a specified frequency domain[4].

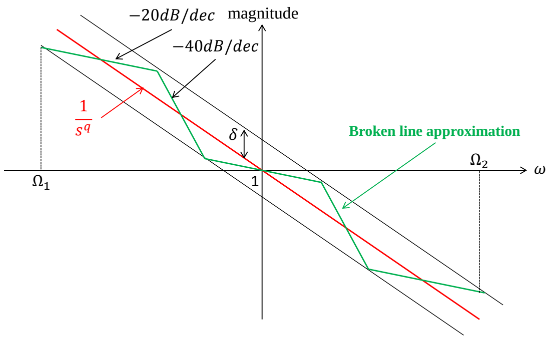

Fig. 3 The approximation of $1/s^q$ at $1<q<2$.

Fig. 3 shows the approximation of $1/s^q$ at $1<q<2$. Using δ shown in Fig. 3, the transfer function of $1/s^q$ at $1<q<2$ can be approximated to

$$\frac{1}{s^q} \approx \frac{1}{s}\cdot\prod_{i=1}^{j}\frac{s+a_i}{s+b_i}\cdot\prod_{i=1}^{k}\frac{1+b_i s}{1+a_i s}, \qquad \Omega_1 \le \omega \le \Omega_2, \tag{5}$$

where

$$\delta = 20\log_{10}\alpha \tag{6}$$

$$\beta = \alpha^{-\frac{2}{(2-q)(q-1)}} \tag{7}$$

$$a_1 = \alpha^{-\frac{1}{q-1}} \tag{8}$$

$$a_{i+1} = a_i\beta \tag{9}$$

$$b_i = a_i\alpha^{-\frac{2}{2-q}} \tag{10}$$

$$\Omega_1 = a_{j+1} \tag{11}$$

$$\Omega_2 = \frac{1}{a_{k+1}} \tag{12}$$

$\Omega_1 \le \omega \le \Omega_2$ is the approximated frequency domain.

In the case of $1/s^\gamma$ at $0<\gamma<1$, the approximated transfer function can be obtained by multiplying Eq. (5) by *s*. It becomes

$$\frac{1}{s^\gamma} = \frac{1}{s^q}\cdot s \approx \prod_{i=1}^{j}\frac{s+a_i}{s+b_i}\cdot\prod_{i=1}^{k}\frac{1+b_i s}{1+a_i s} \qquad 1<q<2 \tag{13}$$

where $\gamma \equiv q-1$.

3 MRACS

3.1 Model Reference Adaptive Control System

Model reference adaptive control system (MRACS) is the control method tuning the controller to match the controlled system containing unknown parameters with the designed reference model adaptively.

Consider the single input single output, continuous time, linear, and time invariant system;

$$y(t) = G(s)u(t); \tag{14}$$

$$G(s) \equiv kB(s)/A(s) = k\left(s^m + \sum_{i=0}^{m-1} b_i s^i\right)\Big/ s^n + \sum_{i=0}^{n-1} a_i s^i, \tag{15}$$

where a_i and b_i are unknown parameters.

Based on this controlled object, we designed the following reference model;

$$y_M(t) = G_M(s)r(t); \tag{16}$$

$$G_M(s) \equiv k_M B_M(s)/A_M(s), = k_M\left(s^{m_M} + \sum_{i=0}^{m_M-1} b_{Mi} s^i\right)\Big/ s^{n_M} + \sum_{i=0}^{n_M-1} a_{Mi} s^i, \tag{17}$$

where a_{Mi} and b_{Mi} are designed parameters.

For the plant Eq. (14) and the reference model Eq. (16), we make the following assumptions:

(A1) The order of controlled object n and relative order of controlled object $n-m$ are already known and $\rho = n-m \le n_M - m_M$.

(A2) $B(s)$ is a stable polynomial.

(A3) The sign of the high frequency gain k is known a priori.

(A4) Without loss of generality, we assume $k>0$.

To guarantee the uniqueness of the solution of the system controlled adaptively for matching the plant's output $y(t)$ with the reference model's output y_M (t), the following lemma has been used.

Lemma 1: Considering ρ order monic polynomial $C(s)$ and $n-1$ order monic polynomial $H(s)$, $R(s)$, $R_B(s)$ and $S(s)$ satisfying following equations are uniquely determined.

$$C(s)H(s) = A(s)R(s) + kS(s) \tag{18}$$

$$R_B(s) = R(s)B(s) - H(s) \tag{19}$$

This is called “Model matching conditions.” In this paper, $C(s)$ and $H(s)$ are designed, and $R(s)$ and $S(s)$ are tuned adaptively. Fig. 4 shows the construction of the system satisfying the model matching condition.

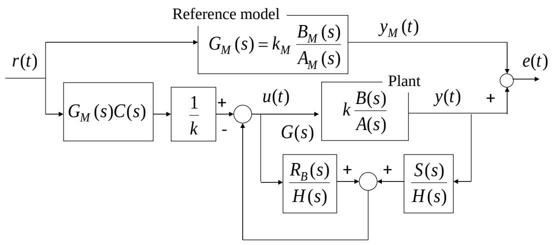

Fig. 4 Construction of the system satisfying the model matching condition.

When the model matching condition is satisfied, the control input $u(t)$ can be expressed as following equation.

$$\begin{aligned} u(t) &= \frac{1}{k}G_M(s)C(s)r(t) - \frac{R_B(s)}{H(s)}u(t) - \frac{S(s)}{H(s)}y(t) \\ &= \frac{1}{k}G_M(s)C(s)r(t) - \sum_{i=0}^{n-2} r_{Bi}\frac{s^i}{H(s)}u(t) - \sum_{i=0}^{n-1} s_i \frac{s^i}{H(s)}y(t) \\ &= \theta^T \varsigma(t) \end{aligned} \tag{20}$$

where we define unkown parameters and regressive vector as follows.

$$\theta \equiv \left[\frac{1}{k}, r_{B0}, \cdots, r_{B(n-2)}, s_0, \cdots, s_{n-1}\right]^{\mathrm{T}} \in \Re^{2n} \tag{21}$$

$$\varsigma(t) \equiv \left[G_M C r, \frac{-1}{H}u, \cdots, \frac{-s^{n-2}}{H}u, \frac{-1}{H}y, \cdots, \frac{-s^{n-1}}{H}y\right]^{\mathrm{T}} \tag{22}$$

And then, the output of the controlled object $y(t)$ can be described as follows.

$$y(t) = \frac{k}{C(s)}\left[u(t) + \frac{R_B(s)}{H(s)}u(t) + \frac{S(s)}{H(s)}y(t)\right] \tag{23}$$

Defining the tracking error as $e(t) = y(t) - y_M(t)$,

$$\begin{aligned} e(t) &= \frac{k}{C(s)}\left[u(t) + \frac{R_B(s)}{H(s)}u(t) + \frac{S(s)}{H(s)}y(t)\right] - y_M(t) \\ &= \frac{k}{C(s)}\left[u(t) + \frac{R_B(s)}{H(s)}u(t) + \frac{S(s)}{H(s)}y(t) - \frac{1}{k}C(s)G_M(s)r(t)\right] \\ &= \frac{k}{C(s)}\left[u(t) - \theta^{\mathrm{T}}\varsigma(t)\right] \end{aligned} \tag{24}$$

The parameters of controlled object is unknown, so we can not use θ as control input. By replacing unkown paremeters with adaptive parameters law, control input $u(t)$ is constructed as following equation.

$$u(t) = \hat{\theta}^{\mathrm{T}}(t)\varsigma(t) \tag{25}$$

Substituting Eq. (24) for Eq. (25), $e(t)$ is replaced with the following equation,

$$e(t) = \frac{k}{C(s)}\left[\tilde{\theta}^{\mathrm{T}}(t)\varsigma(t)\right] \tag{26}$$

where $\tilde{\theta}(t)$ is the error of the parameter difined as $\tilde{\theta}(t) \equiv \hat{\theta}(t) - \theta$. If $\rho = 1$, with using

$$\dot{\tilde{\theta}}(t) = -\mathrm{sgn}(k)\Gamma\varsigma(t)e(t), \quad \Gamma = \Gamma^T > 0 \tag{27}$$

as the adaptive law estimating parameters, boundedness of all signals in the system and global asymptotic stability of the tracking error can be guaranteed. In Eq. (27), the adaptive transfer function uses integer order integration as $1/s$. Fig. 5 shows the construction of the MRACS.

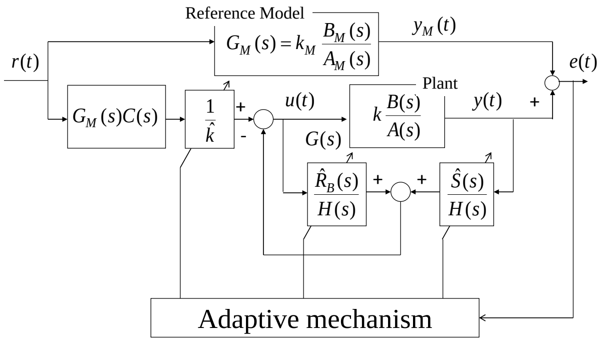

Fig. 5 Construction of the MRACS

3.2 Adaptive control law using fractional integrator

In this paper, we propose a new adaptive law using fractional calculus replacing integer order integration with fractional order integration as

$$\tilde{\theta}^{(q)}(t) = -\mathrm{sgn}(k)\Gamma\varsigma(t)e(t), \quad \Gamma = \Gamma^T > 0 \tag{28}$$

In Eq. (28), the adaptive transfer function shown in Eq. (27) is replaced with $1/s^q$.

Fig. 6 shows the Nyquist plot of the ideal 1/2 order integrator.

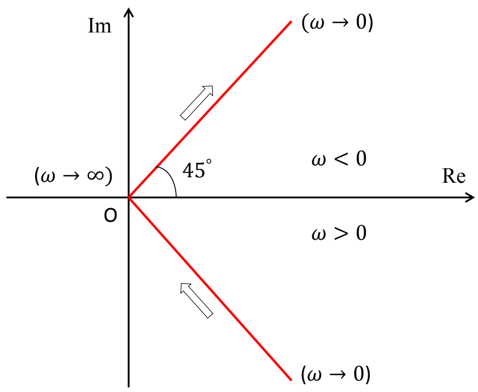

Fig. 6 The Nyquist plot of the ideal 1/2 order integrator.

This transfer function can be seemed to be the integrator which has the infinite gain in the low frequency region. In addition to this characteristic, this transfer function can set a limit to the phase shift less than $\pm 90^\circ$.

So, replacing the integer order integrator with the fractional order integrator, the phase margin can be guaranteed, the problem from the relative order can be eased, and the transient behavior can be improved.

3.3 MRACS for the fractional calculus system

Fractional calculus can descrive the complex system like visco-elastic body’s response. In this paper, we propose a MRACS for the system descrived by fractional calculus.

Considering the following controlled object containing fractional order derivative.

$$y(t) = G(w)u(t); \tag{29}$$

$$G(w) \equiv kB(w)/A(w) = k\left(w^m + \sum_{i=0}^{m-1} b_i w^i\right) \Big/ w^n + \sum_{i=0}^{n-1} a_i w^i; \tag{30}$$

$$w \equiv s^{1/\mu}, \tag{31}$$

where w means the operation of fractional order derivative and this plant satisfies assumptions descrived as (A1) – (A4).

Based on this controlled object, we designed the following reference model extended to the fractional order system;

$$y_M(t) = G_M(w)r(t); \tag{32}$$

$$G_M(w) \equiv k_M B_M(w)/A_M(w),$$

$$= k_M\left(w^{m_M} + \sum_{i=0}^{m_M-1} b_{Mi} w^i\right) \Big/ w^{n_M} + \sum_{i=0}^{n_M-1} a_{Mi} w^i, \tag{33}$$

where a_{Mi} and b_{Mi} are designed parameters.

By replacing s with w, model matching condition can be expressed as follow.

Lemma 2: Considering ρ order monic polynomial $C(w)$ and $n-1$ order monic polynomial $H(w)$, $R(w)$, $R_B(w)$ and $S(w)$ satisfying following equations are uniquely determined.

$$C(w)H(w) = A(w)R(w) + kS(w) \tag{34}$$

$$R_B(w) = R(w)B(w) - H(w) \tag{35}$$

By designing appropriate $C(w)$ and $H(w)$, we can construct the MRACS for the fractional calculus system.

4 NUMERICAL SIMULATION

4.1 Numerical simulation of the approximated fractional integrator

We calculated the fractional order integrator (1/2 order integrator) approximated by Manabe's approach as q = 1.5, δ = 3, j = k = 1 for implementation. We obtained the following function.

$$\frac{1}{s^{0.5}} \approx \frac{s+0.501}{s+0.126} \cdot \frac{1+0.126s}{1+0.501s}, \qquad 0.032 \le \omega \le 31.6, \tag{29}$$

Fig. 7 shows the numerical simulation of the Bode diagram of 1/2 order integrator $(1/s^{0.5})$ approximated by Manabe's approach.

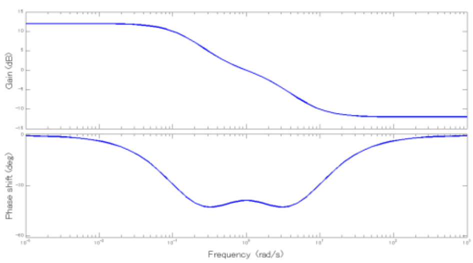

Fig. 7 Bode diagram of approximated 1/2 order integrator.

As shown in Fig. 7, this approximated fractional order integrator has the characteristic of the $1/s^{0.5}$ at $0.032 \le \omega \le 31.6$.

Fig. 8 shows the numerical simulation of its Nyquist plot.

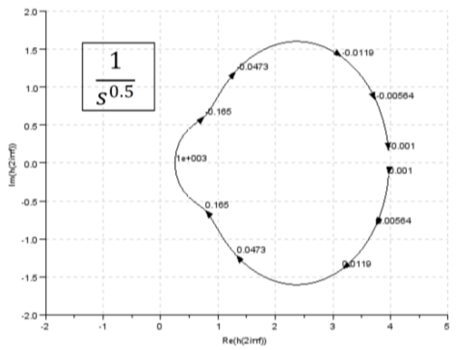

Fig. 8. Nyquist plot of approximated 1/2 order integrator.

The phase change of this approximated 1/2 order integrator is limited in ±38.7° and its gain is 12.0 dB at $s = 0$. The approximated fractional integrator's gain at $s = 0$ can be increased by increasing the order of the integrator. $(j+k)$ Calculating the approximated 1/2 order integrator as j = k = 5 and δ = 3, its gain at $s = 0$ became about 60 dB.

4.2 Numerical simulation of the fractional order MRACS

To show the effectiveness of the adaptive law using the fractional order integrator, we compared the integer order adaptive transfer function $(q=1)$ with fractional order adaptive transfer function $(q=0.5)$ by simulating in following two cases. Fig. 9 shows the construction of the fractional order MRACS designed for the numerical simulation.

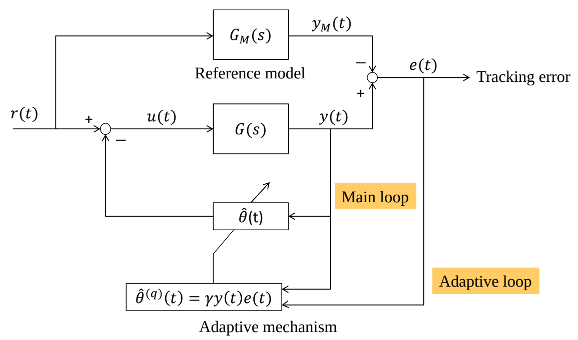

Fig. 9 Construction of the fractional order MRACS designed for the numerical simulation.

Case1: Satisfying model matching condition.

$$G(s) = 1/(s+0.5),\ G_M(s) = 1/(s+1),\ \Gamma = 1000,\ r(t) = \begin{cases} 1 & 1 < t \\ 0 & t < 1 \end{cases}$$

Tracking errors and control input of $q=1$ and $q=0.5$ were shown in Fig. 10 and Fig. 11.

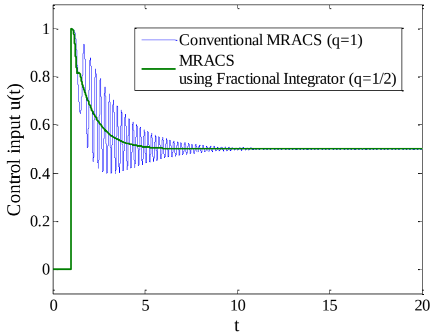

Fig.10 The tracking errors in case1.

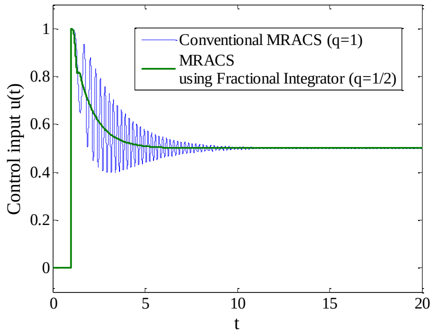

Fig.11 Control input in case1.

As shown in Fig. 10, by using fractional calculus in the adaptive transfer function, the the transient response of the control system was improved.

Case2: Dissatisfying model matching condition.

$G(s)=1/(s+0.5), G_M(s)=1/(s+1)^2, \Gamma=1000$, $r(t)=\sin t$

Tracking errors and control input of $q=1$ and $q=0.5$ were shown in Fig. 12 and Fig. 13.

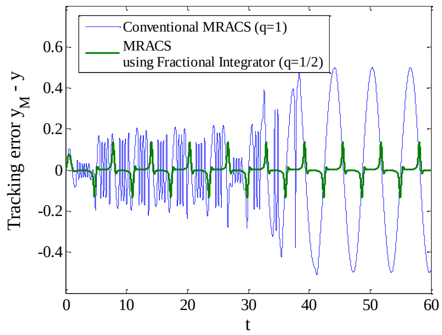

Fig.12 The tracking errors in case2.

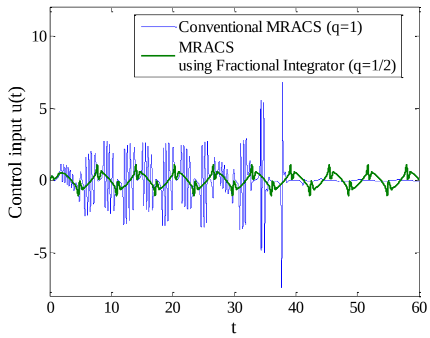

Fig.13 The tracking errors in case2.

As shown in Fig. 12, the response of the system was unstable in $q=1$. However, in $q=0.5$, the internal signal of the system was stabilized without satisfying model matching condition.

4.3 Numerical simulation of MRACS for fractional calculus system

To show the effectiveness of MRACS for the system containing fractional order derivative, we constructed the MRACS and did the numerical simulation in case

$$G(s)=\frac{1}{s^{\frac{1}{2}}+0.5};\quad G_M(s)=\frac{1}{s+1.5};\quad C(s)=1;$$

$$H(s)=1;\ \Gamma=10;\ q=1;\quad r(t)=\begin{cases}1, & 1\le t\\ 0, & t<1\end{cases}$$

In Fig. 16, we showed controlled response of fractional calculus system.

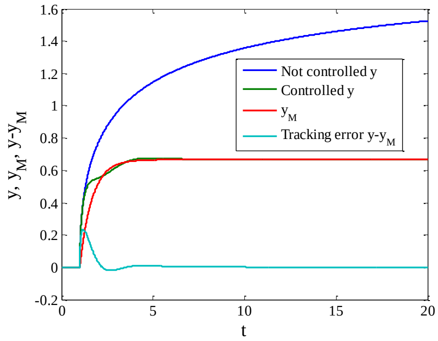

Fig. 16 output of fractional order plant

As shown in Fig. 16, by designing appropriate $C(w)$ and $H(w)$, fractional calculus system can be controlled by MRACS extended to fractional order derivative.

5 Conclusion

In this paper, we applied the fractional calculus to the MRACS. By applying fractional integration to adaptive transfer function, we can improve the transient response and stabilize the system dissatisfying model matching condition. In addition, by extending MRACS to the fractional order derivative, we can construct MRACS for the system containing fractional order derivative.

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