

# Design of Fractional Order Sliding Mode Controller via Non-integer Order Backstepping

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**Abstract:** Novel sliding mode control design is proposed for a fractional order plant in the presence of disturbance and nonlinear terms on mismatched input channel. This mismatched-channel situation is overcome by using the backstepping method. To authors best knowledge, this is first time to present the non-integer order backstepping approach. In order to demonstrate our design efficiency, some numerical simulations are shown.

**Keywords:** Backstepping, Fractional calculus, Non-integer order calculus, Sliding mode control

## 1. INTRODUCTION

Fractional calculus is the operation expanding the order of differential and integral operation from integer to non-integer order. By using fractional calculus, many complicated dynamics like visco-elastic body's response or amorphous semiconductor's electric behavior can be described as the simple system containing fractional order derivative (fractional order system)<sup>[1],[2]</sup>. In addition, fractional calculus is not only useful for the system's expression but useful for the control law. By using fractional calculus in control system, control law containing arbitrary order integration (fractional integrator) can be designed. Fractional integrator has the infinite gain in the low frequency region, and the phase shift of integrator can be limited to less than  $\pm 90^\circ$ , so it is useful for the high gain controller.

Sliding mode control is one of the useful method to control the nonlinear system in the presence of disturbance. However, in the situation of disturbance presence on mismatched input channel, it is needed to design the sliding mode controller through backstepping method<sup>[3]</sup>.

In this paper, we design the sliding mode controller for the fractional order system, and propose the non-integer order backstepping method.

## 2. FRACTIONAL CALCULUS

### 2.1. Fractional Order Rieman-Liouville Integration

Fractional Order Rieman-Liouville Integration is given by

$${}^{RL}_a I_t^q [f(x, t)]_t = \int_a^t \frac{(t - \tau)^{q-1}}{\Gamma(q)} f(x, \tau) d\tau \quad (1)$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (2)$$

where  $q$  is the order of the fractional derivative and  $\Gamma(\cdot)$  is the gamma function, which is the function expanding the factorial to arbitrary order.

### 2.2. Fractional Order Caputo Derivative

Fractional order Caputo derivative<sup>[4]</sup> is given by

$${}_a^C D_t^q [f(x, t)]_t = \int_a^t \frac{(t - \tau)^{n-q-1}}{\Gamma(n - q)} \frac{\partial^n f(x, \tau)}{\partial \tau^n} d\tau \quad (3)$$

where  $q$  is the order of the fractional derivative such that  $n - 1 < q < n$ , and  $n$  is the integer. we defined fractional derivative  $D^q$  as fractional order Caputo derivative ( $D^q \equiv {}_0^C D_t^q$ ).

For example, by using fractional order derivative, we can calculate the  $D^q [t^\nu]$  as

$${}_0^C D_t^q [t^\nu]_t = \int_0^t \frac{(t - \tau)^{n-q-1}}{\Gamma(n - q)} \frac{\partial^n (\tau^\nu)}{\partial \tau^n} d\tau \quad (4)$$

By using Laplace transform  $L \left[ \frac{1}{s^\alpha} \right] = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ , Eq. (4) becomes

$$\begin{aligned} L [{}_0^C D_t^q [t^\nu]_t] &= \frac{1}{s^{n-q}} s^n \frac{\Gamma(\nu + 1)}{s^{\nu+1}} \\ &= \Gamma(\nu + 1) \frac{1}{s^{\nu-q+1}} \end{aligned} \quad (5)$$

From Eq. (5),  $D^q [t^\nu]$  can be described by

$$D^q [t^\nu] = \frac{\Gamma(\nu + 1)}{\Gamma(\nu - q + 1)} t^{\nu-q} \quad (6)$$

As shown in Eq. (6), power function differentiated by fractional Caputo's derivative become the function filling the gap of the  $\nu$ th power function and  $\nu - 1$ th power function.

Next, to show the validness of the definition of Eq. (3), considering the following functions

$$f_1(t) = \sin(t), \quad -\infty < t < \infty \quad (7)$$

$$f_2(t) = \begin{cases} t^2, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (8)$$

${}_{-\infty}^C D_t^q [f_1(t)]$  and  ${}_{-\infty}^C D_t^q [f_2(t)]$  were shown in Fig. 1 and Fig. 2 with changing the order of the fractional derivative  $q$ . In Fig. 1, fractional order Caputo derivative expresses the phase shift derived from the derivation of the sinusoidal function analogically. This feature has the important meaning. By Fourier transform, any

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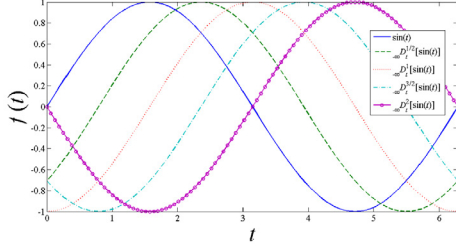


Fig. 1 The function of  ${}_{-\infty}^C D_t^q[f_1(t)]$

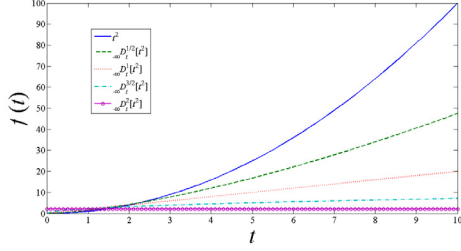


Fig. 2 The function of  ${}_{-\infty}^C D_t^q[f_2(t)]$

function can be converted to the frequency domain spectrum. Considering the derivative as the operation to shift the sinusoidal functions' phase with multiplying its angular frequency, it means any function can be analogically differentiated by fractional order Caputo derivative. As shown in Fig. 2, the fractional order Caputo derivative expresses analogical derivative of  $f_2(t)$  though it is not sinusoidal function.

### 2.3. Chain-rule of Fractional Order Derivative

Fractional order derivative does not obey the same rule of integer order derivative.

For example, chain-rule of integer order derivative can be described by

$$\frac{\partial f(u(t))}{\partial t} = \frac{\partial f(u)}{\partial u} \frac{\partial u}{\partial t} \quad (9)$$

However, if the order of derivative of Eq. (9) is changed to fractional order  $q$ . Because of  $(du)^q \neq d^q u$ , it is obvious that

$$\frac{\partial^q f(u(t))}{\partial t^q} \neq \frac{\partial^q f(u)}{\partial u^q} \frac{\partial^q u}{\partial t^q} \quad (10)$$

So, it is needed to define the chain-rule expanded from to non-integer order.

G. Jumarie show chain-rule of fractional order derivative as following equation<sup>[5]</sup>.

Lemma 1: Assume that both  $f(u)$  and  $u(t)$  are  $q$ th differentiable with respect to  $u$  and  $t$  respectively, then one has the equality

$$\frac{\partial^q f(u(t))}{\partial t^q} = \Gamma(2-q)u^{q-1} \frac{\partial^q f(u)}{\partial u^q} \frac{\partial^q u(t)}{\partial t^q} \quad (11)$$

**Proof:** The  $q$ th derivative of  $u$  with respect to  $u$  is given by following equation.

$$\frac{\partial^q u}{\partial u^q} = \frac{\Gamma(2)}{\Gamma(2-q)}u^{1-q} = \frac{1}{\Gamma(2-q)}u^{1-q} \quad (12)$$

From Eq. (12),  $(\partial u)^q$  can be expressed as

$$(\partial u)^q = \Gamma(2-q)u^{q-1} \partial^q u \quad (13)$$

By using Eq. (12),  $\frac{\partial^q f(u(t))}{\partial t^q}$  becomes

$$\begin{aligned} \frac{\partial^q f(u(t))}{\partial t^q} &= \frac{\partial^q f(u)}{\partial u^q} \frac{\partial^q u}{\partial t^q} \\ &= \Gamma(2-q)u^{q-1} \frac{\partial^q f(u)}{\partial u^q} \frac{\partial^q u}{\partial t^q} \end{aligned} \quad (14)$$

From Eq. (12), following theorem can be get.

**Theorem 1:** If chain-rule of fractional derivative is given by Eq. (12), and if  $V(x)$ ,  $x(t)$ ,  $P$  are given as

$$V(x) = x^\top(t)Px(t) \quad (15)$$

$$x(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)]^\top \quad (16)$$

$$P = \text{diag}[p_1, p_2, \dots, p_n] \quad (17)$$

$q$ th order derivative of  $V(x)$  of  $t$  can be given by following equation.

$$\begin{aligned} \frac{\partial^q V}{\partial t^q} &= \frac{2}{2-q} x^\top(t)P \frac{\partial^q x}{\partial t^q} \\ &= \frac{2}{2-q} \frac{\partial^q x^\top}{\partial t^q} P x(t) \end{aligned} \quad (18)$$

**Proof:**  $V(x)$  can be transformed to the following equation.

$$V(x) = \sum_{i=1}^n p_i x_i^2 \quad (19)$$

From Eq. (11),  $\frac{\partial^q V}{\partial t^q}$  becomes

$$\begin{aligned} \frac{\partial^q V}{\partial t^q} &= \sum_{i=1}^n \Gamma(2-q)x_i^{q-1} \frac{\partial^q V}{\partial x_i^q} \frac{\partial^q x_i}{\partial t^q} \\ &= \sum_{i=1}^n \Gamma(2-q)x_i^{q-1} \frac{\partial^q (p_i x_i^2)}{\partial x_i^q} \frac{\partial^q x_i}{\partial t^q} \\ &= \sum_{i=1}^n \Gamma(2-q)p_i x_i^{q-1} \frac{\partial^q (x_i^2)}{\partial x_i^q} \frac{\partial^q x_i}{\partial t^q} \end{aligned} \quad (20)$$

By Eq. (6), Eq. (20) can be get as

$$\begin{aligned} \frac{\partial^q V}{\partial t^q} &= \sum_{i=1}^n \Gamma(2-q)p_i x_i^{q-1} \frac{\Gamma(3)}{\Gamma(3-q)} x_i^{2-q} \frac{\partial^q x_i}{\partial t^q} \\ &= \sum_{i=1}^n \frac{\Gamma(2-q)\Gamma(3)}{\Gamma(3-q)} p_i x_i \frac{\partial^q x_i}{\partial t^q} \\ &= \sum_{i=1}^n \frac{\Gamma(2-q) \cdot 2!}{(2-q)\Gamma(2-q)} p_i x_i \frac{\partial^q x_i}{\partial t^q} \\ &= \sum_{i=1}^n \frac{2}{2-q} p_i x_i \frac{\partial^q x_i}{\partial t^q} \\ &= \frac{2}{2-q} x^\top(t)P \frac{\partial^q x}{\partial t^q} \\ &= \frac{2}{2-q} \frac{\partial^q x^\top}{\partial t^q} P x(t) \end{aligned} \quad (21)$$

### 2.4. Fractional Calculus System

As an example of fractional calculus system, we explain about the dynamical response of visco-elastic body.

The relation between strain and stress was given as the following equation.

$$\sigma(t) = \int_{-\infty}^t g(t-\tau) \frac{d\epsilon(\tau)}{d\tau} d\tau + G_0 \epsilon(t) \quad (22)$$

$$g(t) = \int_0^\infty H(\tau) e^{-t\tau} d\tau \quad (23)$$

where  $H(\tau)$  is relaxation spectrum,  $\sigma(t)$  is the stress,  $\epsilon(t)$  is the strain. According to the Rouse theory,  $H(\tau)$  is given by

$$H(\tau) = R\tau^{-\frac{1}{2}} \quad (24)$$

If  $R$  is defined as

$$R = G_{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)^{-2} \quad (25)$$

Eq. (22) becomes

$$\sigma(t) = G_{\frac{1}{2}} D^{\frac{1}{2}}[\epsilon(t)] + G_0 \epsilon(t) \quad (26)$$

As shown in Eq. (26), the dynamical response of visco-elastic body can be described by fractional calculus. However,  $R$  is the variable impacted by temperature and molecular structure. So,  $G_{1/2}$  is the variable changing with environment. In Fig. 3, the mechanical model of visco-elastic body was shown. (Fractional order Voigt model) Fractional order Voigt model contains spring pot, which is the element producing the stress proportional to the fractional order differential of the strain (like  $d^{1/2}\epsilon/dt^{1/2}$ ).

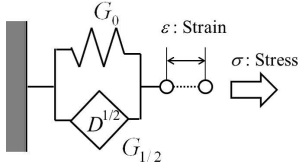


Fig. 3 Fractional order Voigt Model.

## 2.5. Approximation Method of Fractional Calculus

In the case of constructing the control system using the fractional order transfer function, it requires a lot of time to calculate the convolution from the initial time. For reducing the time on calculation, many researchers proposed the approach to approximate the fractional order transfer function to integer order transfer functions<sup>[6],[7],[8]</sup>. We use the approximation method of Manabe<sup>[9]</sup> to calculate the fractional order derivative for the numerical simulation. By using Manabe approach, the transfer function of  $1/s^q$  at  $1 < q < 2$  can be approximated to

$$\frac{1}{s^q} = \frac{1}{s} \cdot \prod_{i=1}^j \frac{s+a_i}{s+b_i} \cdot \prod_{i=1}^j \frac{1+b_i s}{1+a_i s} \quad (27)$$

$$\Omega_{low} < \omega < \Omega_{high} \quad (28)$$

where

$$\delta = 20 \log_{10} \alpha \quad (29)$$

$$\beta = \alpha^{-\frac{2}{(2-q)(q-1)}} \quad (30)$$

$$a_1 = \alpha^{-\frac{1}{q-1}} \quad (31)$$

$$a_{i+1} = a_i \beta \quad (32)$$

$$b_i = a_i \alpha^{-\frac{2}{2-q}} \quad (33)$$

$$\Omega_{low} = \alpha_{j+1} \quad (34)$$

$$\Omega_{high} = \frac{1}{\alpha_{k+1}} \quad (35)$$

$\Omega_{low} < \omega < \Omega_{high}$  is the approximated frequency domain. In the case of  $1/s^r$  at  $0 < r < 1$ , the approximated transfer function can be obtained by multiplying (27) by  $s$ . It becomes

$$\frac{1}{s^r} = \frac{1}{s^q} \cdot s = \prod_{i=1}^j \frac{s+a_i}{s+b_i} \cdot \prod_{i=1}^j \frac{1+b_i s}{1+a_i s} \quad (36)$$

where  $r = 1 - q$ .

## 3. SLIDING MODE CONTROL FOR FRACTIONAL ORDER SYSTEM

Considering the following fractional order system,

$$D^q [x(t)] = f(x) + bu(t) + \delta(t) \quad (0 < q < 1) \quad (37)$$

where  $u(t)$  is control input, and  $\delta(t)$  is disturbance of the system. To design the sliding mode controller for the system described by (37), following switching hyperplane was designed.

$$\sigma = k_p I^{(1-q)} [x(t)] + I^{(2-q)} [x(t)] \quad (38)$$

$$\dot{\sigma} = -w\sigma - k_s \text{sign}(\sigma) \quad (39)$$

$$I^{(\alpha)} [g(t)] = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} g(\tau) d\tau \quad (40)$$

From Eq. (38) - Eq. (40), control input of sliding mode controller  $u_{eq}$  can be calculated as following equation.

$$u_{eq} = -(k_p b)^{-1} \{k_p f(x) + (1 + wk_p) I^{(1-q)} [x(t)]\} - (k_p b)^{-1} \{w I^{(2-q)} [x(t)] + k_s \text{sign}(\sigma)\} \quad (41)$$

Next, considering the following Lyapunov function  $V(t)$ ,

$$V(t) = \frac{1}{2} \sigma^2 \quad (42)$$

From Eq. (42),  $\dot{V}(t)$  becomes

$$\dot{V}(t) = -k_s |\sigma| + k_p \delta(t) \sigma - w \sigma^2 \quad (43)$$

From Eq. (43), if the disturbance of the system  $\delta(t)$  satisfies  $\frac{k_s}{k_p} > |\delta(t)|$ , it becomes  $\dot{V}(t) < 0$ . (Control system is the Lyapunov stable in the case  $\frac{k_s}{k_p} > |\delta(t)|$ .)

## 4. NON-INTEGGER ORDER BACKSTEPPING

Considering the following fractional order system,

$$\begin{aligned} D^q [x_1(t)] &= f_1(x) + g_1(x) x_2 \\ D^q [x_2(t)] &= f_2(x) + g_2(x) x_3 \\ &\vdots \\ D^q [x_{n-1}(t)] &= f_{n-1}(x) + g_{n-1}(x) x_n \\ D^q [x_n(t)] &= f_n(x) + g_n(x) u(t) \end{aligned} \quad (44)$$

Eq. (44) can be described by

$$D^q [x_i(t)] = f_i(\mathbf{x}) + g_i(\mathbf{x})\alpha_i + g_i(\mathbf{x})z_i \quad (45)$$

$$D^q [x_n(t)] = f_n(\mathbf{x}) + g_n(\mathbf{x})u(t) \quad (46)$$

$$z_i = x_{i+1} - \alpha_i \quad (47)$$

$$(i = 1, \dots, n-1)$$

where  $z_i$  is error state, and  $\alpha_i$  is stabilizing function.

To design the control input, considering the following Lyapunov functions,

$$V_0 = \frac{2-q}{2} x_1^2 \quad (48)$$

$$V_i = V_{i-1} + \frac{2-q}{2} z_i^2 \quad (i = 1, \dots, n-1) \quad (49)$$

By using chain rule of fractional order derivative, the  $q$ th order derivative of Eq. (48) and Eq. (49) becomes

$$V_0^{(q)} = x_1 x_1^{(q)} = x_1 (f_1(\mathbf{x}) + g_1(\mathbf{x})\alpha_1) \quad (50)$$

$$V_i^{(q)} = V_{i-1}^{(q)} + z_i z_i^{(q)} = V_{i-1}^{(q)} + z_i (f_{i+1}(\mathbf{x}) + g_{i+1}(\mathbf{x})\alpha_{i+1} - \alpha_i^{(q)}) \quad (51)$$

$$\alpha_n = u(t) \quad (52)$$

$$\alpha_i^{(q)} = \sum_{j=1}^n \Gamma(2-q) x_j^{1-q} \frac{\partial^q \alpha_i}{\partial x_j^q} \frac{\partial^q x_j}{\partial t^q} \quad (53)$$

$$(i = 1, \dots, n-1)$$

If  $\alpha_i$ , ( $i = 1, \dots, n$ ) are designed as  $V_i^{(q)} < 0$ , ( $i = 0, \dots, n-1$ ), control system is stable<sup>[6]</sup>, and desired control input  $u(t)$  can be calculated. By using sliding mode control input Eq. (41) as the stabilizing function  $\alpha_i$ , sliding mode controller can be designed through the backstepping method.

## 5. NUMERICAL SIMULATION

Considering the following fractional order system,

$$D^{\frac{1}{2}} [x_1] = x_2 + 0.5 \sin(100t) \quad (54)$$

$$D^{\frac{1}{2}} [x_2] = u(t) \quad (55)$$

For the system described by Eq. (54) and Eq. (55), we design the sliding mode controller through the backstepping method as

$$\sigma = k_p I^{\frac{1}{2}} [x_1(t)] + I^{\frac{3}{2}} [x_1(t)] \quad (56)$$

$$\dot{\sigma} = -w\sigma - k_s \text{sign}(\sigma) \quad (57)$$

$$\alpha_1 = -x_1 + k_p^{-1} (1 + wk_p) I^{\frac{1}{2}} [x_1(t)] + wk_p^{-1} I^{\frac{3}{2}} [x_1(t)] + k_p^{-1} k_s \text{sign}(\sigma) \quad (58)$$

$$\alpha_2 = -k (x_2(t) - \alpha_1) - x_1 \quad (59)$$

$$k_p = 1 \quad (60)$$

$$k_s = 1 \quad (61)$$

$$w = 1 \quad (62)$$

$$k = \frac{2}{3} \quad (63)$$

In Fig. 4, we show the result of the numerical simulation. As shown in Fig. 4, we can stabilize the system described as Eq. (54) and Eq. (55) by proposed sliding mode controller designed by backstepping.

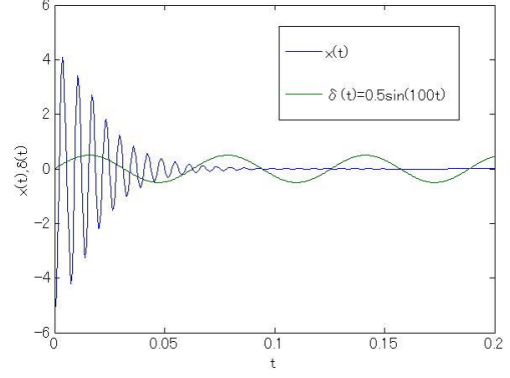


Fig. 4 Disturbance and  $x_1$ .

## 6. CONCLUSION

In this paper, we proposed the non-integer order backstepping method to design the sliding mode controller for the fractional order system on the mismatched-channel situation. And, by numerical simulation, we show the effectiveness of the proposed method.

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