Design of Model Reference Adaptive Control Systems with Fractional Order Adaptive Law and its Lyapunov Stability

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Abstract: Model reference adaptive control systems (MRACS) is the one of the useful systems to control the plant containing unknown variable parameters. However, to design MRACS, it has been needed to satisfy model matching condition. In this paper, to improve the response in the case unsatisfying model matching condition, we design an adaptive law containing fractional order integrator in adaptive law, and prove the system's Lyapnov stability.

Key Words: Fractional Calculus, Non-Integer Order Calculus, Adaptive Control, MRACS, Lyapnov Stability

1 Introduction

Fractional calculus is the operation expanding the order of differential and integral operation from integer to non-integer order. By using fractional calculus, many complicated dynamics like visco-erastic body's responce or amorphous semiconductor's electric behavior can be described as the simple system containing fractional order derivative (fractional order system)[1],[2]. By using fractional calculus in control system, we can design the control law containing arbitrary order integration (fractional integrator.) Fractional integrator has the infinite gain in the low frequency region, and the phase shift of integrator can be limited to less than $\pm 90°$.

Model Reference Control System (MRACS) is the control system tuning its controller to match the controlled system to the designed reference model if plant parameters are unknown[3]. However, to design the MRACS, the assumption of the relative order of the controlled object is needed in order to stabilize the total adaptive control systems. If we design the MRACS without the assumption for the relative order of the controlled object, the performance of control system can be get worse.

By introducing fractional integrator to the adaptive control law in MRACS, it is expected that the performance of the MRACS unsatisfying model matching condition can be improved.

In this paper, we prove the lyapnov stability of MRACS using $1/2$ order integrator through the realization of distributed parameter system, and we show the stable numerical results whose transient performance is good in the case of unsatisfying the model matching condition.

2 Fractional Calculus

2.1 Fractional Order Caputo Derivative

Fractional order Caputo derivative[4] is given by

$$ {}^{C}_{a}D^{q}_{t}[f(t)] = \int_a^t \frac{(t-\tau)^{n-q-1}}{\Gamma(n-q)} \frac{d^n f(\tau)}{d\tau^n} d\tau \tag{1}$$

where q is the order of the fractional derivative such that $n-1 < q < n$, n is the integer, and $\Gamma(\cdot)$ is the gamma function, which is the function expanding the factorial to arbitraly order.we defined fractional derivative D^q as fractional order Caputo derivative $\left(D^q \equiv {}^{C}_{0}D^{q}_{t}\right)$.

2.2 Distributed Parameter System and Fractional Calculus

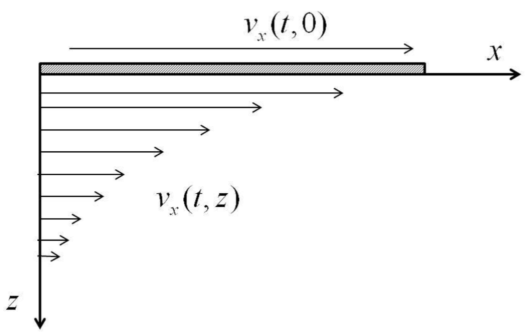

Fig. 1: Rayleigh problem

Considering the system shown in Fig. 1. In this system, lower half part under the plate is filled with Newton-fluid. The motion of the plate and Newton-fluid can be given as the following equations.

$$\frac{\partial v_x}{\partial t}(t,z) = \frac{\eta}{\rho}\frac{\partial^2 v_x}{\partial z^2}(t,z) \tag{2}$$

$$\sigma_{xz}(t,z) = \eta \frac{\partial v_x}{\partial z}(t,z) \tag{3}$$

$$\sigma_{xz}(t,0) = -\sigma_0 \tag{4}$$

$$v_x(t,0) = v_0(t) \tag{5}$$

$$v_x(0,z) = 0 \tag{6}$$

$$\lim_{z\to\infty} v_x(t,z) = 0 \tag{7}$$

where t is time, $v_x(t,z)$ is the x direction of velocity at z position, ρ, η are the density and the viscosity of Newton-fluid, $\sigma_{xz}(t,z)$ is the x direction of shear stress at z position at z position. In this subsection, means the Laplace transformation. By using Laplace transformation and Eq. (5), Eq. (2)

can be transformed to the folloing equation.

$$s\hat{v}_x(s,z) = \frac{\eta}{\rho}\frac{\partial^2 \hat{v}_x}{\partial z^2}(s,z) + v_x(0,z) = \frac{\eta}{\rho}\frac{\partial^2 \hat{v}_x}{\partial z^2}(s,z) \tag{8}$$

so it becomes

$$\hat{v}_x(s,z) = \frac{\eta}{\rho s}\frac{\partial^2 \hat{v}_x}{\partial z^2}(s,z) \tag{9}$$

From Eqs. (5) - (7), the solution of this system can be expressed as follow.

$$\hat{v}_x(s,z) = \hat{v}_0(s) e^{-\sqrt{\frac{\rho}{\eta}} s^{\frac{1}{2}} z} \tag{10}$$

From Eq. (3), $\sigma_{xz}(t,z)$ can be expressed as

$$\hat{\sigma}_{xz}(s,z) = -\sqrt{\rho\eta}\, s^{\frac{1}{2}} \hat{v}_x(s,z) \tag{11}$$

From Eq. (4) and Eq. (5), the x direction of velocity at $z = 0$ position can be expressed as follow.

$$\hat{v}_0(s) = \frac{1}{\sqrt{\rho\eta}}\frac{1}{s^{\frac{1}{2}}}\hat{\sigma}_0(s) \tag{12}$$

As shown in Eq. (12), fractional calculus appears in distributed parameter system.

2.3 Approximation Method of Fractional Calculus

In the case of constructing the control system using the fractional order transfer function, it requires a lot of time to calculate the convolution from the initial time. For reducing the time on calculation, Manabe proposed the approach to approximate the fractional order transfer function to the superposition of the integer order transfer functions on the bode diagram around a specified frequency domain[5].We used this approximation method to calculate the fractional order derivative for the numerical simulation in section 2 and section 8.

Fig. 2 shows the approximation of $1/s^{1.5}$. The transfer function of $1/s^q$ at $1 < q < 2$ can be approximated to

$$\frac{1}{s^q} = \frac{1}{s}\cdot \Pi_{i=1}^{j}\frac{s+a_i}{s+b_i}\cdot \Pi_{i=1}^{j}\frac{1+b_i s}{1+a_i s} \tag{13}$$

$$\Omega_{low} < \omega < \Omega_{high} \tag{14}$$

where

$$\delta = 20\log_{10}\alpha \tag{15}$$

$$\beta = \alpha^{-\frac{2}{(2-q)(q-1)}} \tag{16}$$

$$a_1 = \alpha^{-\frac{1}{q-1}} \tag{17}$$

$$a_{i+1} = a_i\beta \tag{18}$$

$$b_i = a_i\alpha^{-\frac{2}{2-q}} \tag{19}$$

$$\Omega_{low} = \alpha_{j+1} \tag{20}$$

$$\Omega_{high} = \frac{1}{\alpha_{k+1}} \tag{21}$$

$\Omega_{low} < \omega < \Omega_{high}$ is the approximated frequency domain.

In the case of $1/s^r$ at $0 < r < 1$, the approximated transfer function can be obtained by multiplying (13) by s. It becomes

$$\frac{1}{s^r} = \frac{1}{s^q}\cdot s = \Pi_{i=1}^{j}\frac{s+a_i}{s+b_i}\cdot \Pi_{i=1}^{j}\frac{1+b_i s}{1+a_i s} \tag{22}$$

where $r = 1 - q$.

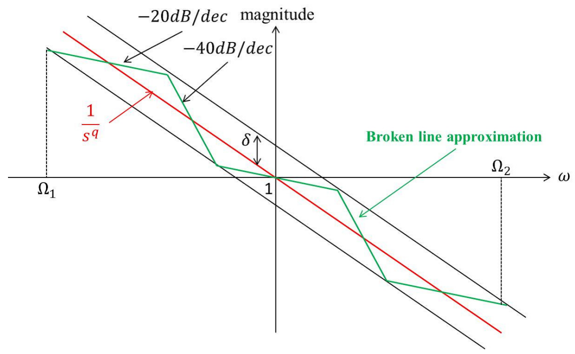

Fig. 2: The approximation of $1/s^{1.5}$

3 Model Reference Adaptive Control System

Considering the following controlled object,

$$y(t) = G(s)u(t) \tag{23}$$

where $y(t)$ is plant's output, $u(t)$ is control input, and $G(s)$ is described as

$$G(s) = \frac{kB(s)}{A(s)} = \frac{k\left(s^m + \sum_{i=0}^{m-1} b_i s^i\right)}{s^n + \sum_{i=0}^{n-1} a_i s^i} \tag{24}$$

To design the controller matching the plant's output $y(t)$ with reference output $y_M(t)$, considering the following reference model

$$y_M(t) = G_M(s)r(t) \tag{25}$$

where

$$G_M(s) = \frac{k_M B_M(s)}{A_M(s)} = \frac{k_M\left(s^{m_M} + \sum_{i=0}^{m_M-1} b_{Mi} s^i\right)}{s^{n_M} + \sum_{i=0}^{n_M-1} a_{Mi} s^i} \tag{26}$$

Here, considering the following assumptions.
(A1) The order of controlled object n and relative order of controlled object $0 < n - m \le n_M - m_M$ is already known.
(A2) Plant's parameters $a_i,\ b_i,\ i = 1,\cdots,n$ is already known.
(A3) $B(s)$ is a stable polynomial.
(A4) Without loss of generality, we assume $k > 0$.

Next, considering the following monic polynominals $C(s)$ and $H(s)$,

$$C(s) = s^{n-m} + \sum_{i=1}^{n-m-1} c_i s^i \tag{27}$$

$$H(s) = s^{n-1} + \sum_{i=1}^{n-2} h_i s^i \tag{28}$$

Then, $R(s)$, $S(s)$, $R_B(s)$ satisfying following equations are uniquely determined. (model matching condition)

$$C(s)H(s) = A(s)R(s) + kS(s) \tag{29}$$

$$R(s) = s^{n-m-1} + \sum_{i=0}^{n-m-2} r_i s^i \tag{30}$$

$$S(s) = s^{n-1} + \sum_{i=0}^{n-2} s_i s^i \tag{31}$$

$$\begin{aligned} R_B(s) &= R(s)B(s) - H(s) \\ &= s^{n-2} + \sum_{i=0}^{n-3} r_{Bi} s^i \end{aligned} \tag{32}$$

Multiplying both sides of Eq. (29) by $y(t)$, following eqation can be given

$$\begin{aligned} y(t) &= \frac{k}{C(s)}\left[u(t) + \frac{R_B(s)}{H(s)}u(t) + \frac{S(s)}{H(s)}y(t)\right] \\ &= kW(s)\left[u(t) + \frac{R_B(s)}{H(s)}u(t) + \frac{S(s)}{H(s)}y(t)\right] \end{aligned} \tag{33}$$

where

$$W(s) \triangleq \frac{1}{C(s)} \tag{34}$$

From Eq. (33) and Eq. (25), $e(t) = y(t) - y_M(t)$ becomes

$$\begin{aligned} e(t) &= y(t) - y_M(t) \\ &= kW(s)\left[u(t) + \frac{R_B(s)}{H(s)}u(t) + \frac{S(s)}{H(s)}y(t)\right] - y_M(t) \\ &= kW(s)\left[u(t) - \theta^\top \zeta(t)\right] \end{aligned} \tag{35}$$

$$\theta = \begin{bmatrix} \frac{1}{k} & \theta_u^\top & \theta_y^\top \end{bmatrix}^\top \tag{36}$$

$$\theta_u = \begin{bmatrix} r_{B0} & \cdots & r_{B(n-2)} \end{bmatrix}^\top \tag{37}$$

$$\theta_y = \begin{bmatrix} s_0 & \cdots & s_{n-1} \end{bmatrix}^\top \tag{38}$$

$$\zeta(t) = \begin{bmatrix} G_M C(s) r(t) & \zeta_u^\top(t) & \zeta_y^\top(t) \end{bmatrix}^\top \tag{39}$$

$$\zeta_u(t) = \begin{bmatrix} \frac{-1}{H(s)}u(t) & \cdots & \frac{-s^{n-2}}{H(s)}u(t) \end{bmatrix}^\top \tag{40}$$

$$\zeta_y(t) = \begin{bmatrix} \frac{-1}{H(s)}y(t) & \cdots & \frac{-s^{n-1}}{H(s)}y(t) \end{bmatrix}^\top \tag{41}$$

Then, if we make the control input $u(t)$ as $u(t) = \theta^\top \zeta(t)$, error of plant's output $y(t)$ and reference output $y_M(t)$ becomes as $e(t) = 0$. However, plant's parameters are unknown. It is needed to construct the control input $u(t)$ by estemated parameters like

$$u(t) = \hat{\theta}^\top(t)\zeta(t) \tag{42}$$

where $\hat{\theta}(t)$ is estimated parameters of θ.

If control input $u(t)$ is given as Eq. (42), output error $e(t) = y(t) - y_M(t)$ becomes

$$e(t) = kW(s)\left[\tilde{\theta}^\top(t)\zeta(t)\right] \tag{43}$$

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta \tag{44}$$

If $W(s)$ is SPR (relative order $n - m = 1$), conventional MRACS uses the following first order integral adaptive law.

$$\dot{\tilde{\theta}}(t) = -\Gamma_\theta \zeta(t) e(t), \quad \Gamma_\theta = \Gamma_\theta^\top > 0 \tag{45}$$

In this paper, we construct the MRACS containing 1/2 order integrator as the adaptive control law.

3.1 1/2 Order Adaptive Control Law and Its Lyapnov Stability

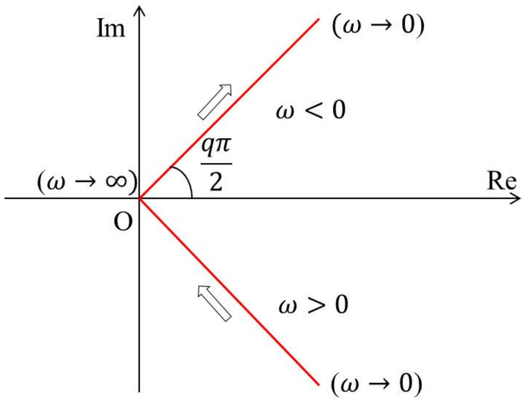

Fig. 3: Nyquist plot of ideal s^{-q} ($q = 1/2$.)

The Nyquist plot of ideal 1/2 order integrator was shown in Fig. 3. As shown in Fig. 3, the phase change of 1/2 order integrator is limited in ±45°, so it is expected that the phase margin of high gain controller can be improved by fractional integrator. Constructing the adaptive control law of MRACS by

$$\tilde{\theta}^{(1/2)}(t) = -\Gamma_\theta \zeta(t) e(t) \tag{46}$$

Then, the construction of MRACS can be shown as Fig. 4.

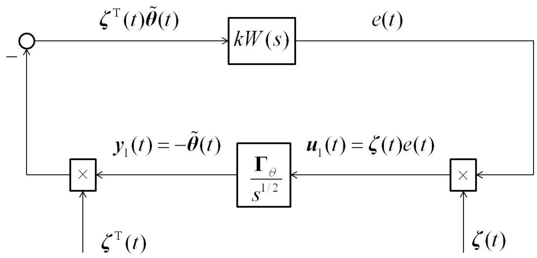

Fig. 4: Construction of MRACS using 1/2 order integrator.

To show the Lyapnov stability of MRACS containing 1/2 order adaptive control law, following lemma was shown.

Lemma 1 （*Kalman-Yakubovich Lemma*）*If and only if $kW(s)$ is SPR, $kW(s)$ has the following realization.*

$$\dot{x}_0(t) = Ax_0(t) + b\left[\tilde{\theta}^\top(t)\zeta(t)\right] \tag{47}$$

$$e(t) = c^\top x_0(t) \tag{48}$$

and there exist positive definite symmetric matrixs P and Q satisfying following equations.

$$AP + PA^\top = -Q \tag{49}$$

$$b^\top P = c^\top \tag{50}$$

Next, we show the following theorem as the realization of 1/2 order system.

Theorem 1 *If there is the 1/2 order system expressed as*

$$\hat{y}_1(s) = \frac{\Gamma_\theta}{s^{1/2}}\hat{u}_1(s) \tag{51}$$

This system has the following realization

$$\frac{\partial^2 x_1}{\partial z^2} = \mu \frac{\partial x_1}{\partial t} \quad (52)$$

$$x_2(t,z) = -\mu^{-1/2}\Gamma_\theta^{-1}\frac{\partial x_1}{\partial z} \quad (53)$$

$$\frac{\partial^2 x_2}{\partial z^2} = \mu \frac{\partial x_2}{\partial t} \quad (54)$$

$$y_1(t) = x_1(t,0) \quad (55)$$

$$u_1(t) = x_2(t,0) \quad (56)$$

$$\lim_{z\to\infty} x_1(t,z) = 0 \quad (57)$$

$$\mu > 0 \quad (58)$$

and this system satisfies passivity.

Proof. Considering $\hat{x}_1(s,z)$, $\hat{x}_2(s,z)$ defined as following equations,

$$\hat{x}_1(s,z) = \hat{y}_1(s)e^{-\mu^{1/2}s^{1/2}z} \quad (59)$$

$$\hat{x}_2(s,z) = \Gamma_\theta^{-1}s^{1/2}\hat{x}_1(s,z) \quad (60)$$

$$\mu > 0 \quad (61)$$

From Eq. (59) and Eq. (60), we can get the following equations.

$$\begin{aligned}\frac{\partial^2 \hat{x}_1}{\partial z^2} &= \mu s\hat{y}_1(s)e^{-\mu^{1/2}s^{1/2}z} \\ &= \mu s\hat{x}_1(s,z) \quad (62)\\ \frac{\partial \hat{x}_1}{\partial z} &= -\mu^{1/2}s^{1/2}\hat{y}_1(s)e^{-\mu^{1/2}s^{1/2}z} \\ &= -\mu^{1/2}s^{1/2}\hat{x}_1(s,z) \\ &= -\mu^{1/2}\Gamma_\theta\hat{x}_2(s,z) \quad (63)\\ \frac{\partial^2 \hat{x}_2}{\partial z^2} &= \Gamma_\theta^{-1}s^{1/2}\frac{\partial^2 \hat{x}_1}{\partial z^2} \\ &= \Gamma_\theta^{-1}s^{1/2}\mu s\hat{x}_1(s,z) \\ &= \Gamma_\theta^{-1}s^{3/2}\mu\hat{x}_1(s,z) \\ &= \mu s\hat{x}_2(s,z) \quad (64)\\ \hat{x}_1(s,0) &= \hat{y}_1(s) \quad (65)\\ \hat{x}_2(s,0) &= \Gamma_\theta^{-1}s^{1/2}\hat{x}_1(s,0) \\ &= \Gamma_\theta^{-1}s^{1/2}\hat{y}_1(s) \\ &= \hat{u}_1(s) \quad (66)\end{aligned}$$

By using inverse Laplace transform, Eq. (59) and Eq. (61), $\hat{y}_1(s) = \frac{\Gamma_\theta}{s^{1/2}}\hat{u}_1(s)$ can be realized by

$$\frac{\partial^2 x_1}{\partial z^2} = \mu \frac{\partial x_1}{\partial t} \quad (67)$$

$$\frac{\partial x_1}{\partial z} = -\mu^{1/2}\Gamma_\theta x_2(t,z) \quad (68)$$

$$\frac{\partial^2 x_2}{\partial z^2} = \mu \frac{\partial x_2}{\partial t} \quad (69)$$

$$y_1(t) = x_1(t,0) \quad (70)$$

$$u_1(t) = x_2(t,0) \quad (71)$$

$$\lim_{z\to\infty} x_1(t,z) = 0 \quad (72)$$

$$\mu > 0 \quad (73)$$

Next, we show the passivity of realized system. Considering the following Lyapnov function.

$$V_1(t,x_1) = \int_0^\infty \frac{1}{2}x_1^\top \rho x_1 \mathrm{d}z, \rho > 0 \quad (74)$$

Then, the first derivative of $V_1(t,x_1)$ can be given as

$$\begin{aligned}\frac{\mathrm{d}V_1}{\mathrm{d}t} &= \int_0^\infty x_1^\top\rho\dot{x}_1\mathrm{d}z = \int_0^\infty x_1^\top\rho\mu^{-1}\frac{\partial^2 x_1}{\partial z^2}\mathrm{d}z \\ &= \int_0^\infty x_1^\top\rho\mu^{-1}\frac{\partial}{\partial z}\left(-\mu^{1/2}\Gamma_\theta x_2\right)\mathrm{d}z \\ &= -\int_0^\infty \mu^{-1/2}x_1^\top\rho\Gamma_\theta\frac{\partial x_2}{\partial z}\mathrm{d}z \\ &= -\mu^{-1/2}\left[x_1^\top\rho\Gamma_\theta x_2\right]_0^\infty + \int_0^\infty \mu^{-1/2}\frac{\partial x_1^\top}{\partial z}\rho\Gamma_\theta x_2\mathrm{d}z \\ &= -\mu^{-1/2}\left[x_1^\top\rho\Gamma_\theta x_2\right]_0^\infty \\ &\quad + \int_0^\infty \mu^{-1/2}\left(-\mu^{1/2}x_2^\top\Gamma_\theta^\top\right)\rho\Gamma_\theta x_2\mathrm{d}z \\ &= \mu^{-1/2}\lim_{z\to 0}\left[x_1^\top\rho\Gamma_\theta x_2\right] - \int_0^\infty x_2^\top\Gamma_\theta^\top\rho\Gamma_\theta x_2\mathrm{d}z \\ &= \mu^{-1/2}y_1^\top(t)\rho\Gamma_\theta u_1(t) - \int_0^\infty x_2^\top\Gamma_\theta^\top\rho\Gamma_\theta x_2\mathrm{d}z \quad (75)\end{aligned}$$

If ρ becomes

$$\rho = \mu^{1/2}\Gamma_\theta^{-1} > 0 \quad (76)$$

Eq. (75) can be discribed by

$$\begin{aligned}\frac{\mathrm{d}V_1}{\mathrm{d}t} &= y_1^\top(t)u_1(t) - \int_0^\infty x_2^\top\Gamma_\theta^\top\rho\Gamma_\theta x_2\mathrm{d}z \\ &= y_1^\top(t)u_1(t) - \alpha_1^2 \le y_1^\top(t)u_1(t) \quad (77)\end{aligned}$$

Q.E.D.

From Lemma 1 and Theorem 1, following theorem can be shown. If the error transfer function is expressed as Eq. (43) and $W(s)$ is SPR, MRACS using following adaptive law is Lyapnov stable.

$$\tilde{\theta}^{\left(\frac{1}{2}\right)}(t) = -\Gamma_\theta\zeta(t)e(t) \quad (78)$$

Proof. From Lemma 1, the realization of $kW(s)$ can be given as Eq. (47) and Eq. (48). And, $\frac{\Gamma_\theta}{s^{1/2}}$ can be realized by the following equations from Theorem 1.

$$\frac{\partial^2 x_1}{\partial z^2} = \mu \frac{\partial x_1}{\partial t} \quad (79)$$

$$\frac{\partial x_1}{\partial z} = -\mu^{1/2}\Gamma_\theta x_2(t,z) \quad (80)$$

$$\frac{\partial^2 x_2}{\partial z^2} = \mu \frac{\partial x_2}{\partial t} \quad (81)$$

$$-\tilde{\theta}(t) = x_1(t,0) \quad (82)$$

$$\zeta(t)e(t) = x_2(t,0) \quad (83)$$

$$\lim_{z\to\infty} x_1(t,z) = 0 \quad (84)$$

$$\mu > 0 \quad (85)$$

Considering the following Lyapnov function,

$$V(t) = \frac{1}{2}x_0^\top(t)Px_0 + \int_0^\infty x_1^\top(t)\rho x_1(t)\mathrm{d}z \quad (86)$$

By using (75), the first derivative of Eq. (86) becomes

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \frac{1}{2}\dot{x}_0^{\top}(t)Px_0(t)+\frac{1}{2}x_0^{\top}(t)P\dot{x}_0(t)\\ &\quad -\mu^{-1/2}\frac{1}{2}\tilde{\theta}^{\top}(t)\rho\Gamma_{\theta}\zeta(t)e(t)\\ &\quad -\int_0^{\infty}x_2^{\top}(t)\Gamma_{\theta}^{\top}\rho\Gamma_{\theta}x_2\mathrm{d}z\\ &= x_0^{\top}(t)\left(PA+A^{\top}P\right)x_0(t)\\ &\quad +\tilde{\theta}^{\top}(t)\zeta(t)b^{\top}Px_0(t)\\ &\quad -\mu^{-1/2}\frac{1}{2}\tilde{\theta}^{\top}(t)\rho\Gamma_{\theta}\zeta(t)e(t)\\ &\quad -\int_0^{\infty}x_2^{\top}(t)\Gamma_{\theta}^{\top}\rho\Gamma_{\theta}x_2\mathrm{d}z\\ &= x_0^{\top}(t)\left(PA+A^{\top}P\right)x_0(t)+\tilde{\theta}^{\top}(t)\zeta(t)ke(t)\\ &\quad -\mu^{-1/2}\frac{1}{2}\tilde{\theta}^{\top}(t)\rho\Gamma_{\theta}\zeta(t)e(t)\\ &\quad -\int_0^{\infty}x_2^{\top}(t)\Gamma_{\theta}^{\top}\rho\Gamma_{\theta}x_2\mathrm{d}z \end{aligned} \tag{87}$$

If ρ becomes

$$\rho=\mu^{-1/2}\Gamma_{\theta}^{-1}>0 \tag{88}$$

and $kW(s)$ is SPR, we can get the following equation.

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= -x_0^{\top}(t)Qx_0(t)\\ &\quad -\int_0^{\infty}x_2^{\top}(t)\Gamma_{\theta}^{\top}\rho\Gamma x_2\mathrm{d}z<0 \end{aligned} \tag{89}$$

Q.E.D.

4 Numerical Simulation

In this section, we construct the MRACS shown as Fig. 5 and calculate the approximated 1/2 order integrator as shown in Fig. 6 by Manabe approach.

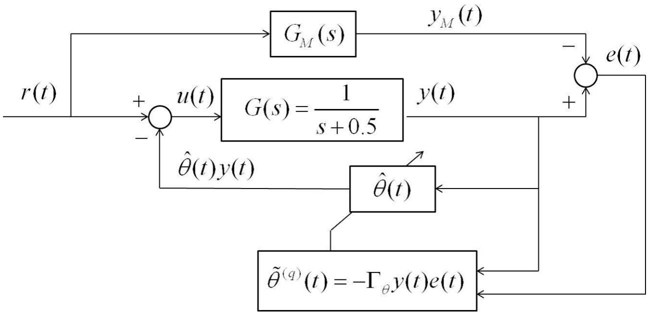

Fig. 5: Construction of MRACS used in numerical simulation.

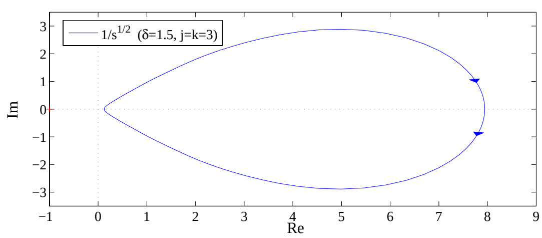

Fig. 6: Approximated 1/2 order integrator.

To show the effectiveness of the adaptive law using the fractional order integrator, we compared the integer order adaptive transfer function ($q=1$) with with fractional order adaptive transfer function ($q=1/2$) by simulating in two cases.

Case 1. Satisfying model matching condition.

$$G(s) = \frac{1}{s+0.5} \tag{90}$$

$$G_M(s) = \frac{1}{s+1} \tag{91}$$

$$\Gamma_{\theta} = 1000 \tag{92}$$

$$r(t) = \begin{cases}1, & t\geq 1\\ 0, & t<1\end{cases} \tag{93}$$

Tracking errors, control input and estimated parameter of $q=1$ and $q=1/2$ in case 1 were shown in Fig. 7, Fig. 8 and Fig. 9. As shown in Fig. 7, Fig. 8 and Fig. 9, by using the fractional integrator, transient response of MRACS using high gain adaptive law can be improved.

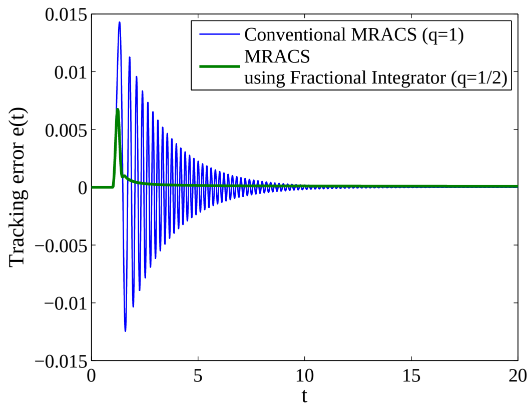

Fig. 7: Tracking error $e(t)=y-y_M$ in Case 1.

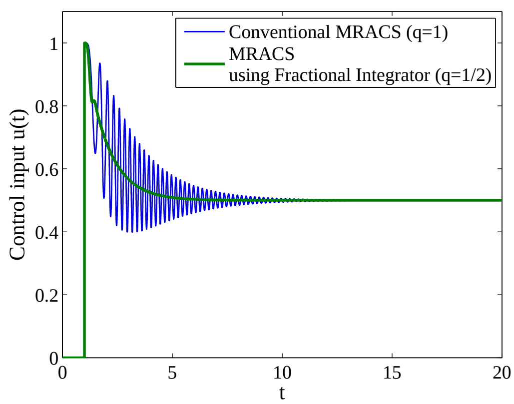

Fig. 8: Control input $u(t)$ in Case 1.

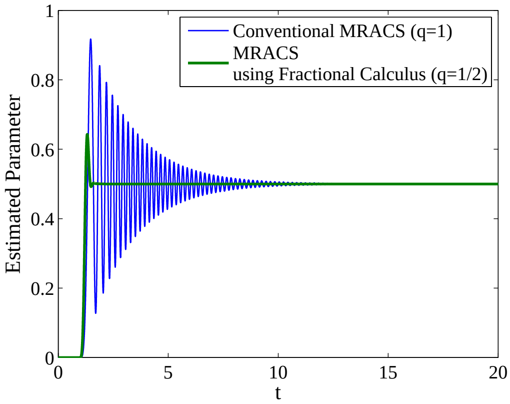

Fig. 9: Estimated parameter $\hat{\theta}(t)$ in Case 1.

Case 2. Unsatisfying model matching condition.

$$G(s) = \frac{1}{s+0.5} \tag{94}$$

$$G_M(s) = \frac{1}{(s+1)^2} \tag{95}$$

$$\Gamma_\theta = 1000 \tag{96}$$

$$r(t) = \begin{cases} \sin(t), & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{97}$$

Tracking errors, control input and estimated parameter of $q = 1$ and $q = 1/2$ in case 2 were shown in Fig. 10, Fig. 11 and Fig. 12. As shown in Fig. 10, Fig. 11 and Fig. 12, the response of the MRACS has unstable in $q = 1$. However, in $q = 1/2$, the internal signal of the system was stabilized without satisfying model matching condition.

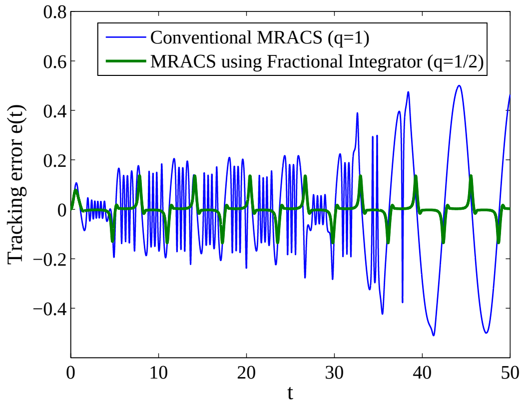

Fig. 10: Tracking error $e(t) = y - y_M$ in Case 2.

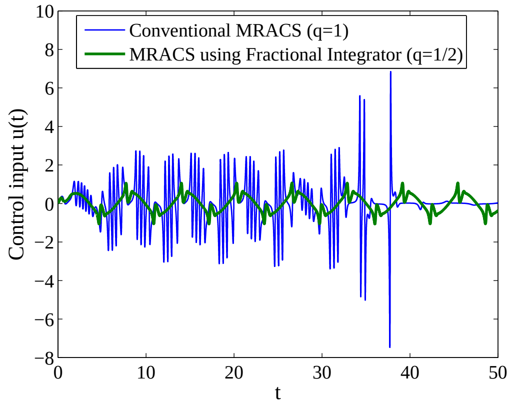

Fig. 11: Control input $u(t)$ in Case 2.

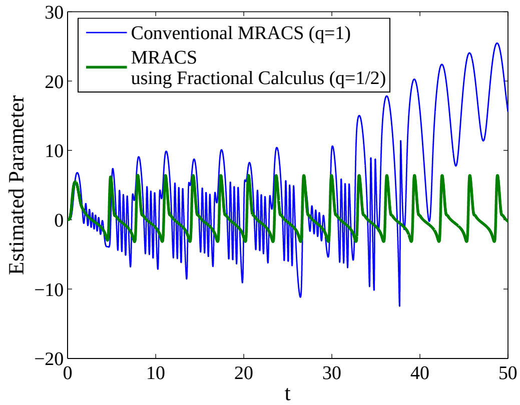

Fig. 12: Estimated parameter $\hat{\theta}(t)$ in Case 2.

5 Conclusion

In this paper, we applied the fractional calculus to the MRACS and prove the Lyapnov stability of MRACS using 1/2 order adaptive law. By applying fractional integration to adaptive transfer function, we can improve the transient response and stabilize the system dissatisfying model matching condition. However, in this paper, we prove only the stability of MRACS using 1/2 order adaptive law in the case relative order $n - m = 1$. As the future work, it is needed to prove the stability of MRACS containg arbitraly order q adaptive law in the case relative order $n - m = 2 - q$.

References

[1] R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, Journal of Rheology Vol. 27, pp. 201-210, 1983.

[2] M. Fukunaga, Application of fractional diffusion equation to amorphous semiconductors, In J. C. Trigeassou A. Le Mehaute, J. A. T. Machado and J. Sabatier, editors, Fractional Differentiation and its Applications, pp. 389-400, Ubooks, 2005.

[3] R. V. Monopoli, Model Reference Adaptive Control with an Augmented Error Signal, IEEE Transactions on Automatic Control, Vol. 19(5), pp. 474-484, 1974.

[4] M. Caputo and F.Mainardi, A Dissipation Model Based on Memory Mechanism, Pure and Applied Geophiscs, Vol. 91(8), pp.134-147, 1971.

[5] S. Manabe, About the control system containing non-integer order integration, Journal of Institute of Electrical Engineers of Japan, 80-5, 589-597, 1980, [In Japanese.]